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COMPUTING AN ALMOST MINIMUM SET OF SPANNING LINE SEGMENTS OF A POLYHEDRON

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ABSTRACT

A *set of spanning line segments* (SLS) is a subset of the edges of a finite polyhedron in E^3 such that an arbitrary plane intersects the polyhedron if and only if the plane intersects at least one of the line segments of the SLS. In this paper an algorithm is presented for computing an almost minimum set of spanning line segments for an arbitrary polyhedron $\mathcal{P}$. When the number of extreme vertices of $\mathcal{P}$ is odd, the computed SLS is minimum; when the number of extreme vertices of $\mathcal{P}$ is even, the size of the computed SLS is at most the minimum size plus one. The algorithm has linear-time complexity for a convex polyhedron, hence is optimal in this case; its time complexity is $\Theta(m \log m)$ for an arbitrary polyhedron, where m is the number of vertices of the polyhedron.

Keywords: Polyhedron, intersection, collision detection, line segment.

1. Introduction

Linear separability is an important problem in pattern recognition for classification. [1] The problem is to decide whether there is a hyperplane separating two sets of points in E^d, $d \geq 1$. This problem has also been studied in computational geometry. [2] In this paper we study an inseparable set of line segments, called a *set of spanning line segments*, in E^3 associated with a polyhedron.

If a plane intersects a polyhedron in E^3, then it intersects one of the edges of the polyhedron. A *set of spanning line segments* (SLS) is a subset of the edges of a polyhedron such that an arbitrary plane intersects the polyhedron if and only if the plane intersects one of the line segments in the SLS. Testing for intersection between a plane and a polyhedron is a fundamental problem and has practical applications. An efficient method based on the set of spanning line segments for testing whether a box intersects a given plane is described in Ref. [3].

An SLS of a polyhedron is, in general, not unique. For instance, all edges of a polyhedron form an SLS of the polyhedron, although an SLS of a smaller size usually exists. Let CH($\mathcal{P}$) denote the convex hull of polyhedron $\mathcal{P}$ in E^3. A vertex A of $\mathcal{P}$ is called an *extreme vertex* if there is a supporting plane of CH($\mathcal{P}$) through A such that A is the only point common to the supporting plane and CH($\mathcal{P}$). It can be shown that, for a polyhedron $\mathcal{P}$ with n extreme vertices, the minimum size of an SLS of $\mathcal{P}$ is $(n+1)/2$ when n is odd, and $n/2$ or $(n/2)+1$ when n is even. Figure 1 shows an hour-glass shaped polyhedron with 20 edges. Its minimum SLS has four line segments, shown in the dotted lines.

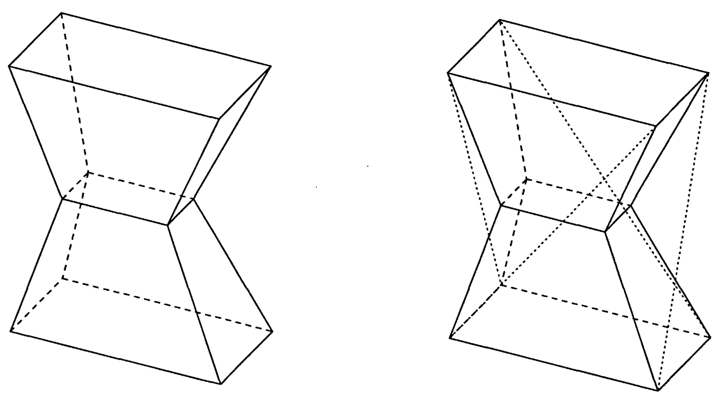

Fig. 1. A polyhedron of 20 edges has an SLS of 4 line segments.

We provide an efficient algorithm to compute an almost minimum SLS of an arbitrary but finite polyhedron in E^3. Let $\mathcal{P}$ be a polyhedron that has n extreme vertices. When n is odd, the algorithm computes an SLS of $(n+1)/2$ line segments, which is minimum. When n is even, the algorithm computes an SLS of $n/2$ or $(n/2)+1$ line segments; the SLS may not be minimum when it has $(n/2)+1$ line segments.

The algorithm works as follows. First, the convex hull of the polyhedron is computed. Then two particular vertices of the convex hull are removed in each step to give a new line segment that is added to a growing set that finally becomes an SLS of the polyhedron. This step is repeated recursively until six or fewer vertices of the convex hull are left, a case that can be resolved directly. The key lies in how to choose the two vertices in each step; the choice must ensure that the set of line segments formed at last is an SLS of the polyhedron.

The time complexity of the algorithm is linear for a convex polyhedron, hence is optimal in this case. When the polyhedron is arbitrary, the convex hull needs to be found first, so the time complexity in this case is $\Theta(m \log m)$, where m is the number of vertices of the polyhedron. The algorithm uses linear space.

In the following, we will study the properties of SLS in section 2 and present our algorithm in section 3.

2. Definition and Properties of SLS

Let $\mathcal{P}$ be a finite polyhedron in E^3. We assume throughout that all the vertices of CH($\mathcal{P}$) are the extreme vertices of $\mathcal{P}$.

Definition 1. *A set $\mathcal{S}$ of line segments is* inseparable *if an arbitrary plane either intersects a line segment in $\mathcal{S}$ or has all the line segments in $\mathcal{S}$ on the same side of the plane.*

Definition 2. *A set $\mathcal{S}$ of line segments is a* set of spanning line segments *(SLS) of a polyhedron $\mathcal{P}$ if $\mathcal{S}$ satisfies the following three conditions:*

(1) The endpoints of each line segment in $\mathcal{S}$ are extreme vertices of $\mathcal{P}$.

(2) Each extreme vertex of $\mathcal{P}$ is an endpoint of some line segment in $\mathcal{S}$.

(3) $\mathcal{S}$ is inseparable.

Let $|\mathcal{S}|$ denote the size of $\mathcal{S}$. The next lemma follows from condition 2 above.

Lemma 1 *Let $\mathcal{P}$ be a polyhedron that has n extreme vertices. Let $\mathcal{S}$ be an SLS of $\mathcal{P}$. Then $|\mathcal{S}| \geq \lceil n/2 \rceil$*

Lemma 2 *Let $\mathcal{P}$ be a polyhedron. A plane intersects $\mathcal{P}$ if and only if the plane intersects* CH($\mathcal{P}$).

The proof of Lemma 2 is straightforward and so is omitted. Lemma 2 implies that a polyhedron can be replaced by its convex hull for computing an SLS of the polyhedron.

Theorem 1 *Let $\mathcal{S}$ be an SLS of a polyhedron $\mathcal{P}$. A plane intersects $\mathcal{P}$ if and only if the plane intersects a line segment of $\mathcal{S}$.*

Proof. Suppose that a plane intersects polyhedron $\mathcal{P}$. Then, by Lemma 2, it intersects CH($\mathcal{P}$). Therefore the plane either passes through a vertex of CH($\mathcal{P}$) or separates the vertices of CH($\mathcal{P}$) into two groups. In the first case, by condition 2 in Definition 2, the plane intersects a member of $\mathcal{S}$. In the second case, since every vertex of CH($\mathcal{P}$) is covered by some line segment of $\mathcal{S}$ and because of the inseparability of $\mathcal{S}$, the plane must intersect a member of $\mathcal{S}$.

Conversely, if a plane intersects a line segment of $\mathcal{S}$, by condition 1 in Definition 2, it intersects CH($\mathcal{P}$). Hence, by Lemma 2, the plane intersects $\mathcal{P}$. □

Let CH($\mathcal{S}$) denote the convex hull of a set $\mathcal{S}$ of line segments.

Lemma 3 *Let $\mathcal{R}$ and $\mathcal{S}$ be two inseparable sets of line segments in E^3. $\mathcal{T} = \mathcal{R} \bigcup \mathcal{S}$ is inseparable if and only if* $\text{CH}(\mathcal{R}) \bigcap \text{CH}(\mathcal{S}) \neq \emptyset$.

Proof. If $\mathcal{T}$ is separable, since $\mathcal{R}$ and $\mathcal{S}$ are inseparable, there exists a plane that separates $\mathcal{R}$ and $\mathcal{S}$. So the $\text{CH}(\mathcal{R}) \bigcap \text{CH}(\mathcal{S}) = \emptyset$. Hence, $\text{CH}(\mathcal{R}) \bigcap \text{CH}(\mathcal{S}) \neq \emptyset$ implies that $\mathcal{T}$ is inseparable.

Conversely, if $\mathrm{CH}(\mathcal{R}) \bigcap \mathrm{CH}(\mathcal{S}) = \emptyset$, there is a plane separating $\mathrm{CH}(\mathcal{R})$ and $\mathrm{CH}(\mathcal{S})$. Therefore the plane separates the sets $\mathcal{R}$ and $\mathcal{S}$, *i.e.*, $\mathcal{T}$ is separable. Hence $\mathrm{CH}(\mathcal{R}) \bigcap \mathrm{CH}(\mathcal{S}) \neq \emptyset$ if $\mathcal{T} = \mathcal{R} \bigcup \mathcal{S}$ is inseparable. □

Next we discuss the SLS of a polyhedron with five or six extreme vertices. The discussion will not only reveal and exemplify some properties of SLS, but also provide the foundation of our algorithm for computing an almost minimum SLS of a general polyhedron.

Lemma 4 *Let $\mathcal{V}$ be the set of vertices of a polyhedron. Let A and B be two distinct extreme vertices of* $\mathrm{CH}(\mathcal{V})$ *such that the line segment (A, B) is not an edge of* $\mathrm{CH}(\mathcal{V})$, *which is treated as a polyhedron. Then (A, B) intersects* $\mathrm{CH}(\mathcal{V} - \{A, B\})$.

Proof. The proof is by contradiction. Suppose that the line segment (A, B) does not intersect $\mathrm{CH}(\mathcal{V} - \{A, B\})$. We first show that the straight line through A and B, denoted by AB, does not intersect $\mathrm{CH}(\mathcal{V} - \{A, B\})$ either. As (A, B) does not intersect $\mathrm{CH}(\mathcal{V} - \{A, B\})$, if the straight line AB intersects $\mathrm{CH}(\mathcal{V} - \{A, B\})$, the intersection must occur on the straight line AB but outside the line segment (A, B). But this implies that at least one of A and B is not an extreme vertex of $\mathrm{CH}(\mathcal{V})$. This is a contradiction. Hence, the straight line AB does not intersect $\mathrm{CH}(\mathcal{V} - \{A, B\})$. Consequently, there exist two distinct supporting planes of $\mathrm{CH}(\mathcal{V} - \{A, B\})$ that pass through the line AB and sandwich $\mathrm{CH}(\mathcal{V} - \{A, B\})$ between them, as illustrated in Figure 2. So (A, B) is an edge of $\mathrm{CH}(\mathcal{V})$. But this contradicts our assumption. Hence, (A, B) intersects $\mathrm{CH}(\mathcal{V} - \{A, B\})$. □

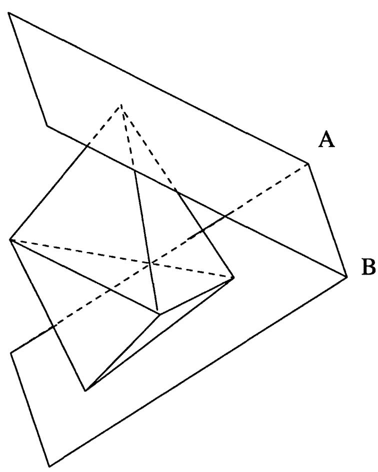

Fig. 2. Two supporting planes of $\mathrm{CH}(V - \{A, B\})$ pass through AB.

Lemma 5 *For a convex polyhedron $\mathcal{P}$ with five vertices there exist two of the five vertices whose connecting line segment is not an edge of $\mathcal{P}$.*

Proof. Suppose that all pairs of vertices of $\mathcal{P}$ form the edges of $\mathcal{P}$. Then $\mathcal{P}$ has $e = \binom{5}{2} = 10$ edges. On the other hand, by Euler's formula, $\mathcal{P}$ has $f = 2 - v + e = 7$ faces. Since each face has at least three sides, the number of edges of $\mathcal{P}$ is $e \geq$

$3 \times 7/2 = 21/2 > 10$. This is a contradiction. Hence, the connecting line segment of some two vertices of $\mathcal{P}$ is not an edge. □

Lemma 5 is reminiscent of the fact that the complete graph K_5 is nonplanar.

Lemma 6 *Let $\mathcal{S}$ be a set of line segments. Suppose* $\mathrm{CH}(\mathcal{S}') \bigcap \mathrm{CH}(\mathcal{S} - \mathcal{S}') \neq \emptyset$ *for any nonempty proper subset $\mathcal{S}' \subset \mathcal{S}$. Then $\mathcal{S}$ is inseparable.*

Proof. If $\mathcal{S}$ is separable, then there is a plane separating $\mathcal{S}$ into two nonempty subsets $\mathcal{S}'$ and $\mathcal{S} - \mathcal{S}'$. It follows that $\mathrm{CH}(\mathcal{S}') \bigcap \mathrm{CH}(\mathcal{S} - \mathcal{S}') = \emptyset$, which is a contradiction. □

Theorem 2 *A minimum SLS of a polyhedron with five extreme vertices has three line segments.*

Proof. Without loss of generality, assume that $\mathcal{P}$ is a convex polyhedron with five extreme vertices. By Lemma 5, there exists a line segment linking two vertices of $\mathcal{P}$ and not being an edge of $\mathrm{CH}(\mathcal{P})$. Denote this line segment by (A, B), as shown in Figure 3. Choose any two line segments from the sides of the triangle formed by the remaining three vertices, and denote them by (C, D) and (C, E). Then we claim that $\mathcal{S} = \{(A, B), (C, D), (C, E)\}$ is an SLS of $\mathcal{P}$. To see this, firstly, it is obvious that the first two conditions of Definition 2 are satisfied by $\mathcal{S}$. Secondly, by Lemma 4, (A, B) intersects $\mathrm{CH}(\{C, D, E\}) \equiv \mathrm{CH}(\{(C, D), (C, E)\})$. Thus, by Lemma 6, $\mathcal{S}$ is inseparable. Hence, $\mathcal{S}$ is an SLS of $\mathcal{P}$. By Lemma 1, $\mathcal{S}$ is minimum. □

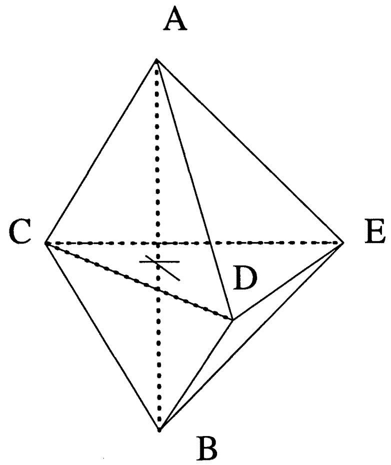

Fig. 3. A polyhedron with 5 extreme vertices has an SLS of 3 line segments.

Lemma 7 *Let $\mathcal{P}$ be a convex polyhedron with n extreme vertices, $n \geq 2$. Suppose that a vertex of $\mathcal{P}$ has degree $n - 1$. Then $|\mathcal{S}| \geq \lceil (n + 1)/2 \rceil$ for any $\mathcal{S}$ that is an SLS of $\mathcal{P}$.*

Proof. Let $\mathcal{V}$ denote the set of vertices of $\mathcal{P}$. Then $|\mathcal{V}| = n$. When n is odd, the lemma is implied by Lemma 1, since $\lceil n/2 \rceil = \lceil (n + 1)/2 \rceil$. Now consider the case where n is even. Let A be a vertex of $\mathcal{P}$ that has degree $n - 1$. Let $\mathcal{S}$ be an SLS of $\mathcal{P}$. By condition 2 in Definition 2, since the degree of A is $n - 1$, $\mathcal{S}$ must contain an edge of $\mathcal{P}$ that has A as one of its endpoints. Denote this edge by (A, B). Then the vertex A or B must be the endpoint of some other line segment, denoted by

(A, C) or (B, C), in $\mathcal{S}$; for otherwise, since (A, B) is an edge of $\mathcal{P}$, (A, B) would be separable from other members of $\mathcal{S}$ by a plane, contradicting that $\mathcal{S}$ is inseparable. Now the remaining $n-3$ vertices, *i.e.*, $\mathcal{V}-\{A, B, C\}$, need $\lceil (n-3)/2 \rceil$ line segments to cover. Hence an SLS of $\mathcal{P}$ has at least $\lceil (n-3)/2 \rceil + 2 = \lceil (n+1)/2 \rceil$ line segments. □

Theorem 3 *For a polyhedron $\mathcal{P}$ with six extreme vertices, its minimum SLS contains three or four line segments. Furthermore, a minimum SLS of $\mathcal{P}$ has four line segments if and only if a vertex of CH($\mathcal{P}$) has degree* 5.

Proof. Without loss of generality, assume that $\mathcal{P}$ is a convex polyhedron and all of its six vertices are extreme. The maximum degree possible of a vertex of $\mathcal{P}$ is 5, and the minimum degree possible is 3. First consider the case where a vertex has degree 5. According to Lemma 7, a minimum SLS of $\mathcal{P}$ has at least $\lceil (6+1)/2 \rceil = 4$ line segments. Now we construct an SLS of $\mathcal{P}$ with four line segments. First we claim that $\mathcal{P}$ has a vertex of degree 3. Otherwise, suppose that all the vertices of $\mathcal{P}$ have degree 4 or above. Suppose further that m vertices of $\mathcal{P}$ have degree 5, where $m \geq 1$. Then the number of edges of $\mathcal{P}$ is $e = [5m + 4 * (6-m)]/2$. Since each face has at least 3 sides, the number of faces of $\mathcal{P}$ is $f \leq 2e/3 = [5m + 4 * (6-m)]/3$. On the other hand, by Euler's formula, $e - f = 4$. It follows that

$$[5m + 4 * (6-m)]/2 - [5m + 4 * (6-m)]/3 \leq 4.$$

The solution to this inequality is $m \leq 0$, which is a contradiction. Therefore $\mathcal{P}$ has a vertex of degree 3. Denote this vertex by B, as shown in Figure 4. Let D and E be two other vertices such that (B, D) and (B, E) are not edges of $\mathcal{P}$. Then we claim that the set $\mathcal{S} = \{(B, D), (B, E), (A, C), (A, F)\}$ is an SLS of $\mathcal{P}$, where A, C, F are the remaining vertices. Since (B, D) is not an edge of $\mathcal{P}$ and the degree of B is 3, D is in the extended cone with apex B and base ΔACF, and D and B on different sides of ΔACF. Therefore, triangle ΔBDE and triangle ΔACF intersect. Hence, by Lemma 6, $\mathcal{S}$ is inseparable.

In all the remaining cases the maximum degree possible of vertices of $\mathcal{P}$ is 4. Let m be the number of the vertices that have degree 4. Then the number of edges of $\mathcal{P}$ is

$$e = [4m + 3 * (6-m)]/2 = (18+m)/2.$$

Since e is an integer, m can only be 0, 2, 4, or 6. In these four cases, by Euler's formula, the number of edges and the number of faces of $\mathcal{P}$ are, respectively, ($m = 0$): $e = 9$, $f = 5$; ($m = 2$): $e = 10$, $f = 6$; ($m = 4$): $e = 11$, $f = 7$; ($m = 6$): $e = 12$, $f = 8$. The configurations of polyhedra satisfying these four sets of conditions are shown in Figure 5. It can be shown through an exhaustive enumeration that each configuration is unique up to the corresponding set of conditions. By verifying the conditions in Definition 2 and applying Lemma 6, it is easy to see that in each case the 3 dotted line segments shown in the figure form an SLS of $\mathcal{P}$. □

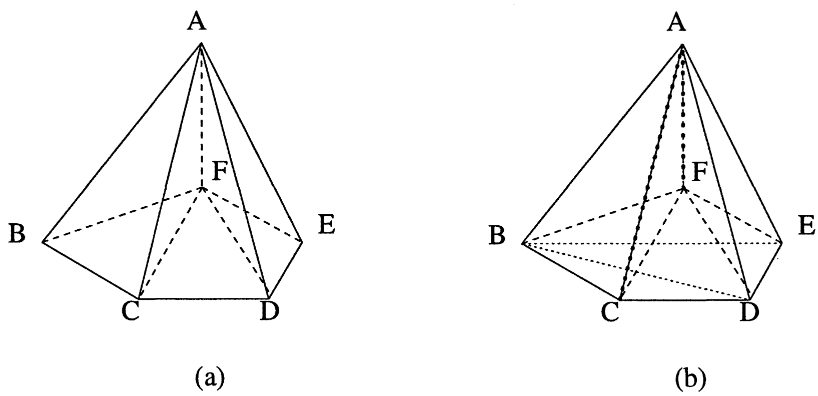

Fig. 4.

For a polyhedron with three or four extreme vertices, an SLS with two or three line segments, respectively, can easily be created. To sum up, for any polyhedron with n extreme vertices, $n \leq 6$, a minimum SLS with $\lceil n/2 \rceil$ or $\lceil n/2 \rceil + 1$ line segments can be computed.

3. The Algorithm

Now we describe the procedure for computing an SLS of a polyhedron.

Procedure FIND-SLS:
Input: A polyhedron $\mathcal{P}$ with $m \geq 3$ vertices.
Output: A set $\mathcal{S}$ of spanning line segments of $\mathcal{P}$.

begin

0. Compute the convex hull CH($\mathcal{P}$) of $\mathcal{P}$ by any standard algorithm;
1. $\mathcal{V}$ = set of vertices of CH($\mathcal{P}$); $\mathcal{S} = \emptyset$;
2. **while**($|\mathcal{V}| > 6$) **do**

   begin

   2.1. Find a subset $\mathcal{V}'$ of five vertices of $\mathcal{V}$;

   2.2. Find two vertices A and B of $\mathcal{V}'$ which do not form an edge of $\mathcal{V}'$;

   2.3. $\mathcal{S} = \mathcal{S} \bigcup \{(A, B)\}$;

   2.4. $\mathcal{V} = \mathcal{V} - \{A, B\}$;

   end
3. Find a minimum SLS $\mathcal{S}_1$ of $\mathcal{V}$; /* $|\mathcal{V}| \leq 6$ in this case. */
4. $\mathcal{S} = \mathcal{S} \bigcup \mathcal{S}_1$;
5. Return;

end

Remarks: The existence of the two vertices A and B selected in step 2.2 is ensured by Lemma 5. The method for finding an SLS of $\mathcal{V}$ when $4 < |\mathcal{V}| \leq 6$ in step 3 is based on the proofs of Theorem 2 and Theorem 3. An SLS of $\mathcal{V}$ with three or four vertices in general positions contains two or three line segments, respectively, and can easily be computed.

The next lemma is straightforward.

Lemma 8 *Let $\mathcal{V}$ be the set of the extreme vertices of a polyhedron. Let $\mathcal{V}'$ be a nonempty subset of $\mathcal{V}$. Then all vertices in $\mathcal{V}'$ are the extreme vertices of* $\mathrm{CH}(\mathcal{V}')$.

Theorem 4 *Let $\mathcal{P}$ be an input polyhedron to* FIND-SLS *that has n extreme vertices. Then the set $\mathcal{S}$ of line segments computed by* FIND-SLS *is an SLS of $\mathcal{P}$. Furthermore, $\mathcal{S}$ is minimum when n is odd, and $|\mathcal{S}|$ may be greater than the minimum size by one when n is even.*

Proof. It is obvious that conditions 1 and 2 in Definition 2 are satisfied by the set of line segments computed. Now we prove its inseparability by induction. First, in step 3, the set of line segments computed for a subset $\mathcal{V}$ with $|\mathcal{V}| \leq 6$ is an SLS of $\mathcal{V}$. When $|\mathcal{V}| > 6$, let $\mathcal{V}'$ be the set of five vertices chosen in the first execution of the while-loop. Let $A, B \in \mathcal{V}'$ be two vertices such that (A, B) is not an edge of $\mathrm{CH}(\mathcal{V}')$. By Lemma 8, $\mathrm{CH}(\mathcal{V}')$ has five extreme vertices, so such vertices A and B exist by Lemma 5.

Let $\mathcal{V}_1 = \mathcal{V} - \{A, B\}$. Suppose that the set of line segments created subsequently for the set $\mathcal{V}_1$ is an SLS of $\mathcal{V}_1$. By the choice of A and B in the while-loop and Lemma 4, the line segment (A, B) intersects $\mathrm{CH}(\mathcal{V}' - \{A, B\})$, and it therefore also intersects $\mathrm{CH}(\mathcal{V}_1)$ since $\mathcal{V}' - \{A, B\} \subset \mathcal{V}_1$. By Lemma 3, the final set $\mathcal{S}$, as the union of $\{A, B\}$ and an SLS of $\mathcal{V}_1$, is also inseparable. Therefore $\mathcal{S}$ is an SLS of $\mathcal{P}$. Hence, by induction, the algorithm computes an SLS of $\mathcal{P}$.

When n is odd, two vertices are removed from $\mathcal{V}$ in each run of the while-loop until there are five vertices left. The set of these five vertices, by Theorem 2, has an SLS containing three line segments. So the SLS computed for $\mathcal{P}$ has $\lceil (n-5)/2 \rceil + 3 = \lceil n/2 \rceil$ line segments, which is minimum, by Lemma 1. When n is even, two vertices are removed from $\mathcal{V}$ each time in the while-loop until six vertices are left, whose minimum SLS, by Theorem 3, has three or four line segments; in this case the SLS computed for $\mathcal{P}$ has $(n-6)/2 + 3 = n/2$ or $(n-6)/2 + 4 = n/2 + 1$ line segments, respectively. It is minimum in the former case but may be minimum plus one in the latter case. □

Theorem 5 *The time complexity of the algorithm* FIND-SLS *is $\Theta(m \log m)$ for an arbitrary polyhedron $\mathcal{P}$, where m is the number of vertices of $\mathcal{P}$. It is linear-time if $\mathcal{P}$ is convex.*

Proof. The time spent on step 0 for finding the convex hull of $\mathcal{P}$ is $\Theta(m \log m)$, and is dominant. When $\mathcal{P}$ is convex, the time on the remaining steps is clearly linear. □

4. Summary

The algorithm FIND-SLS computes a minimum SLS when n is odd, where n is the number of extreme vertices of the input polyhedron. When n is even, the size of the computed SLS may be greater than the minimum size by one. For some polyhedra, including those characterized in Lemma 7, the SLS computed by FIND-SLS is minimum even when n is even. An open problem is to characterize all polyhedra with an even number n of extreme vertices that have a minimum SLS of size $n/2 + 1$.

Another open problem is to design an efficient algorithm to compute a minimum SLS for any polyhedron. For a polyhedron with an even number n of vertices and known to have a minimum SLS of size $n/2$, the algorithm FIND-SLS may still yield an SLS of size $n/2+1$, but a different choice of the two vertices in step 2.2 of the while-loop can lead to a minimum SLS of size $n/2$ of the same polyhedron. For example, in Figure 6, a convex polyhedron with eight extreme vertices is shown. Removing (F, G) first will lead to an SLS of five line segments because the remaining six vertices form a polyhedron with a vertex H having degree 5; this polyhedron, by Theorem 3, has an SLS of four line segments. However, the remaining 6-vertex polyhedron obtained by removing (E, F) first does not have any vertex of degree 5, so its minimum SLS has three line segments by Theorem 3. Thus, the minimum SLS of $\mathcal{P}$ has four line segments.

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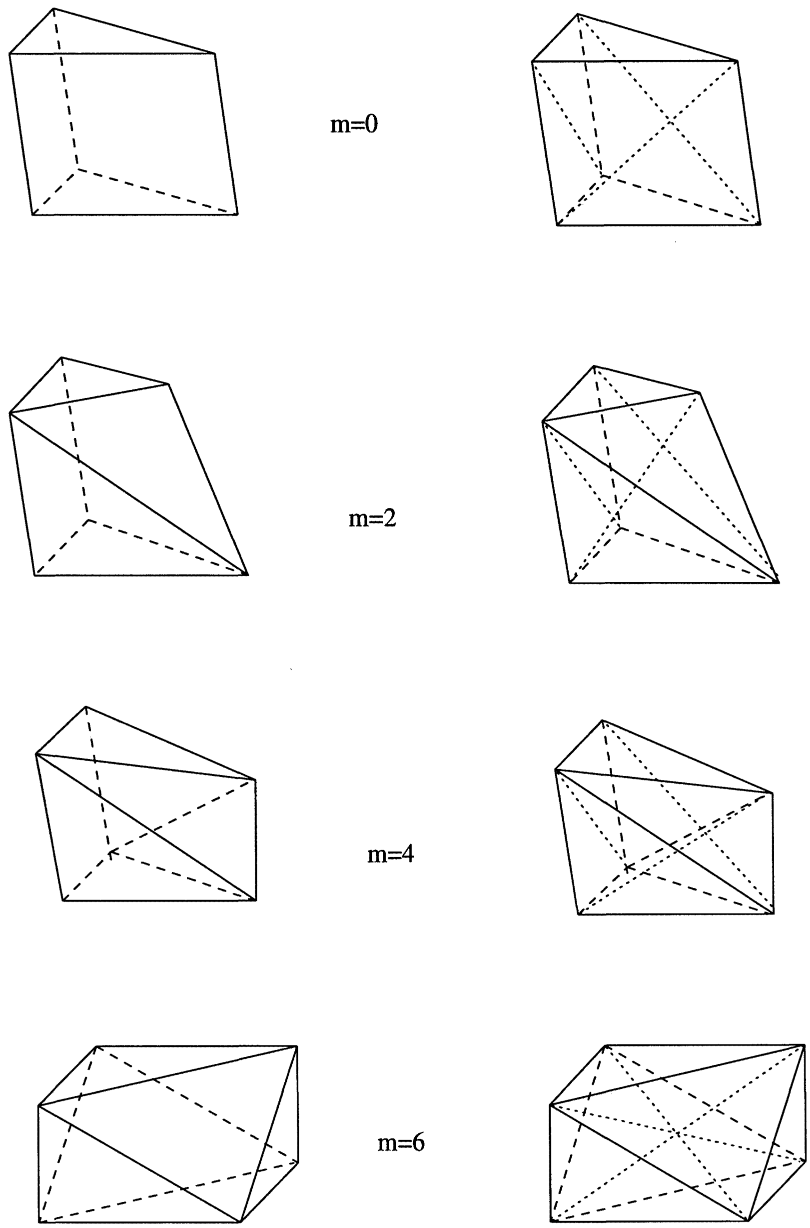

Fig. 5.

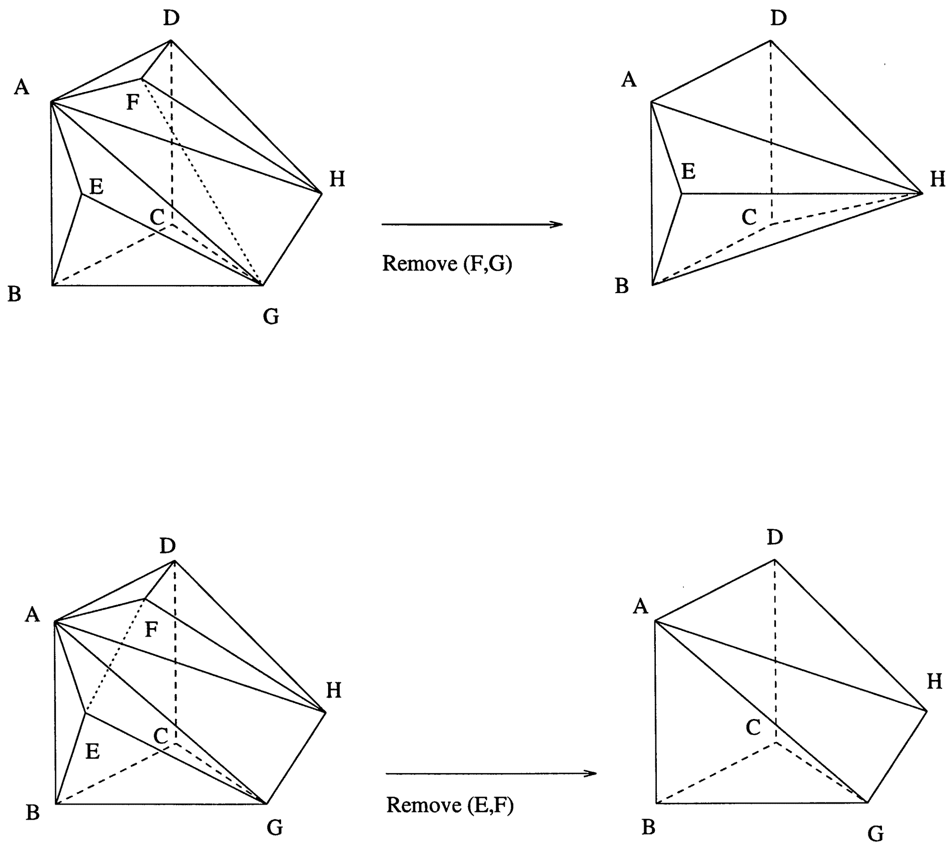

Fig. 6.